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Classification of general Wentzell boundary conditions for fourth order operators in one space dimension

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Abstract

In this paper we consider a fourth order linear ordinary differential operator in one space dimension. We impose, at each endpoint, one general Wentzell boundary condition as well as one other linear boundary. Our goal is to classify precisely when these operators are symmetric, semibounded and/or quasiaccretive. In particular these results extend the collection of boundary conditions for which the one-dimensional beam equation $u_{tt} + c^2 u_{xxxx} = 0$ is well-posed.

Keywords: Wentzell boundary conditions; Dynamic boundary conditions; Selfadjointness; Semiboundedness; Accretive; Ordinary differential operator

1. Introduction

The second order ordinary differential operator $Bu = u''$ acts on functions on the interval $[0, 1]$. The problem of classifying $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $(\beta_1, \beta_2, \beta_3, \beta_4)$ so that the operator B is symmetric, selfadjoint and/or bounded above was considered by Hellwig [12]. Here

$$\alpha_1 u'(0) + \alpha_2 u(0) + \alpha_3 u'(1) + \alpha_4 u(1) = 0 \tag{BC1$'$}$$

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and

$$\beta_1 u'(1) + \beta_2 u(1) + \beta_3 u'(0) + \beta_4 u(0) = 0 \tag{BC2$'$}$$

are the nonseparated boundary conditions for the operator B acting on $\mathcal{H} = L^2(0,1)$ with domain

$$D_0(B) = \{u \in C^2[0,1]: \text{ (BC1}') \text{ and (BC2}') \text{ hold}\}.$$

Consider the operator B equipped with the separated boundary conditions

$$\alpha_0 u''(0) + \alpha_1 u'(0) + \alpha_2 u(0) = 0 \tag{BC1}$$

and

$$\beta_0 u''(1) + \beta_1 u'(1) + \beta_2 u(1) = 0, \tag{BC2}$$

where $(\alpha_0, \alpha_1, \alpha_2)$, $(\beta_0, \beta_1, \beta_2)$ are linearly independent vectors in $\mathbf{R}^3$. When $\alpha_0 = \beta_0 = 0$, then these reduce to the usual Robin (including Dirichlet and Neumann) boundary conditions. When $\alpha_0 = \beta_0 = 1$, the boundary conditions (BC1) and (BC2) are the general Wentzell boundary conditions. It follows from our earlier work [4] that if also $\alpha_1 < 0 < \beta_1$, then B on $\mathcal{H} = L^2(0,1) \oplus \mathbf{C}^2$ is essentially selfadjoint and dissipative on the domain

$$D_1(B) := \{u \in C^2[0,1]: \text{ (BC1) and (BC2) hold}\},$$

provided $\mathcal{H}$ has the norm defined by

$$\|u\|_{\mathcal{H}}^2 = \int_0^1 |u(x)|^2\,dx + \frac{|u(0)|^2}{-\alpha_1} + \frac{|u(1)|^2}{\beta_1}$$

for $u \in C[0,1] \subset \mathcal{H}$. Note that $C[0,1]$, viewed as a subspace of $\mathcal{H}$ as above, is dense, as is $C^\infty[0,1]$. But $\mathcal{H}$ also contains elements of the form (v, z_0, z_1) where $v \in L^2\{0,1\}$, $z_j \in \mathbf{C}$ and z_j is not related to $\lim_{x\to j} v(x)$, even when the latter exists. The classification problem for B with Wentzell boundary conditions (BC1) and (BC2) was considered by Gal [10]; in that paper he also obtained some results for the nonseparated Wentzell boundary conditions

$$\alpha_0 u''(0) + \alpha_1 u'(0) + \alpha_2 u(0) + \alpha_3 u'(1) + \alpha_4 u(1) = 0$$

and

$$\beta_0 u''(1) + \beta_1 u'(1) + \beta_2 u(1) + \beta_3 u'(0) + \beta_4 u(0) = 0.$$

If B is as above and $A = B^2 = u''''$, then A is essentially selfadjoint and nonnegative on $\mathcal{H}$ when the boundary conditions for A are

$$u''(0) + \alpha_2 u'(0) + \alpha_3 u(0) = 0, \qquad u''(1) + \beta_2 u'(1) + \beta_3 u(1) = 0,$$
$$u''''(0) + \alpha_2 u'''(0) + \alpha_3 u''(0) = 0, \qquad u''''(1) + \beta_2 u'''(1) + \beta_3 u''(1) = 0.$$

Surprisingly, if one modifies these boundary conditions (so that $u \to u''''$ is not the square of a second order operator with given boundary conditions), but retains terms of the form

$$u''''(0) + \alpha_2 u'''(0) + \widetilde{\alpha}_3 u''(0) + \widetilde{\alpha}_4 u'(0) + \widetilde{\alpha}_5 u(0) = 0,$$
$$u''''(1) + \beta_2 u'''(1) + \widetilde{\beta}_3 u''(1) + \widetilde{\beta}_4 u'(1) + \widetilde{\beta}_5 u(1) = 0,$$

the symmetry and semiboundedness can be lost. In the following we classify the cases where symmetry and semiboundedness are retained. In Section 5 we show when the associated operator generates a semigroup.

We have included a number of references for the convenience of the reader. For background on semigroups and selfadjoint operators, see [1,2,9,11–13]. For Wentzell boundary conditions for second order operators, see [3–6,14]. For other results on fourth order elliptic operators with general Wentzell boundary conditions, see [7,8].

2. The problem

In this paper we consider the operator

$$Au = u'''' \tag{1}$$

on the interval $[0, 1]$. We consider the boundary conditions

$$u''''(0) + \alpha_3 u'''(0) + \alpha_2 u''(0) + \alpha_1 u'(0) + \alpha_0 u(0) = 0, \tag{2}$$

$$u''''(1) + \beta_3 u'''(1) + \beta_2 u''(1) + \beta_1 u'(1) + \beta_0 u(1) = 0, \tag{3}$$

$$\gamma_3 u'''(0) + \gamma_2 u''(0) + \gamma_1 u'(0) + \gamma_0 u(0) = 0, \tag{4}$$

$$\delta_3 u'''(1) + \delta_2 u''(1) + \delta_1 u'(1) + \delta_0 u(1) = 0. \tag{5}$$

The domain for our operator is

$$\mathcal{D}(A) = \bigl\{u \in C^4[0, 1]\colon\ (2)\text{–}(5)\ \text{hold}\bigr\}.$$

We work in the space $X = L^2[0, 1] \oplus \mathbf{C}^2_w$; the inner product on this space is given by

$$\langle u, v\rangle_X = \langle u, v\rangle = \int_0^1 u(x)\overline{v(x)}\,dx + \sum_{i=0}^{1} u(i)\overline{v(i)}w_i$$

and the norm is determined by

$$|||u|||^2 = \langle u, u\rangle. \tag{6}$$

Here $w_i > 0$ is the weight associated with the endpoint i for $i = 0, 1$.

3. Symmetry conditions

In this section we calculate the conditions on the coefficients $\alpha_i, \beta_i, \gamma_i, \delta_i$ so that our operator is symmetric. Integration by parts gives

$$\begin{aligned}
\langle Au, v\rangle &= \int_0^1 u''''(x)\overline{v(x)}\,dx + \sum_{i=0}^{1} Au(i)\overline{v(i)}w_i \\
&= -\int_0^1 u'''(x)\overline{v'(x)} + u'''(x)\overline{v(x)}\big|_0^1 + \sum_{i=0}^{1} Au(i)\overline{v(i)}w_i \\
&= \int_0^1 u''(x)\overline{v''(x)}\,dx - u''(x)\overline{v'(x)}\big|_0^1 + u'''(x)\overline{v(x)}\big|_0^1 + \sum_{i=0}^{1} Au(i)\overline{v(i)}w_i.
\end{aligned} \tag{7}$$

Similarly,

$$\langle u, Av\rangle = \int_0^1 u''(x)\overline{v''(x)}\,dx - u'(x)\overline{v''(x)}\Big|_0^1 + u(x)\overline{v'''(x)}\Big|_0^1 + \sum_{i=0}^1 u(i)\overline{Av(i)}w_i.$$

Hence, A is symmetric if

$$\begin{aligned}
0 &= \Big[u'(x)\overline{v''(x)}\Big|_0^1 - u''(x)\overline{v'(x)}\Big|_0^1\Big] + \Big[u'''(x)\overline{v(x)}\Big|_0^1 - u(x)\overline{v'''(x)}\Big|_0^1\Big] \\
&\quad - \big[\alpha_3 u'''(0) + \alpha_2 u''(0) + \alpha_1 u'(0) + \alpha_0 u(0)\big]\overline{v(0)}w_0 \\
&\quad + \big[\alpha_3\overline{v'''(0)} + \alpha_2\overline{v''(0)} + \alpha_1\overline{v'(0)} + \alpha_0\overline{v(0)}\,\big]u(0)w_0 \\
&\quad - \big[\beta_3 u'''(1) + \beta_2 u''(1) + \beta_1 u'(1) + \beta_0 u(1)\big]\overline{v(1)}w_1 \\
&\quad + \big[\beta_3\overline{v'''(1)} + \beta_2\overline{v''(1)} + \beta_1\overline{v'(1)} + \beta_0\overline{v(1)}\,\big]u(1)w_1.
\end{aligned} \tag{8}$$

Set $u^{(j)}(i) = u_i^j$ and $\overline{v^{(j)}(i)} = v_i^j$. Next we *choose*

$$w_0 = -\frac{1}{\alpha_3} > 0, \qquad w_1 = \frac{1}{\beta_3} > 0. \tag{9}$$

Thus, we henceforth *assume*

$$\alpha_3 < 0 < \beta_3.$$

Then (8) becomes

$$\begin{aligned}
0 &= \big(u_1^1 v_1^2 - u_1^2 v_1^1\big) - \big(u_0^1 v_0^2 - u_0^2 v_0^1\big) - \alpha_2 w_0\big(u_0^2 v_0^0 - u_0^0 v_0^2\big) - \alpha_1 w_0\big(u_0^1 v_0^0 - u_0^0 v_0^1\big) \\
&\quad - \beta_2 w_1\big(u_1^2 v_1^0 - u_1^0 v_1^2\big) - \beta_1 w_1\big(u_1^1 v_1^0 - u_1^0 v_1^1\big),
\end{aligned}$$

or, equivalently,

$$0 = \det\begin{bmatrix} \beta_3 w_1 & -\beta_2 w_1 & \beta_1 w_1 \\ u_1^0 & u_1^1 & u_1^2 \\ v_1^0 & v_1^1 & v_1^2 \end{bmatrix} - \det\begin{bmatrix} -\alpha_3 w_0 & \alpha_2 w_0 & -\alpha_1 w_0 \\ u_0^0 & u_0^1 & u_0^2 \\ v_0^0 & v_0^1 & v_0^2 \end{bmatrix}. \tag{10}$$

Define

$$B_1 = \begin{bmatrix} \beta_3 w_1 & -\beta_2 w_1 & \beta_1 w_1 \\ u_1^0 & u_1^1 & u_1^2 \\ v_1^0 & v_1^1 & v_1^2 \end{bmatrix} \quad\text{and}\quad B_0 = \begin{bmatrix} -\alpha_3 w_0 & \alpha_2 w_0 & -\alpha_1 w_0 \\ u_0^0 & u_0^1 & u_0^2 \\ v_0^0 & v_0^1 & v_0^2 \end{bmatrix}.$$

Then (8) is equivalent to (10) if $\det B_1 = \det B_0$. Hence we have proved

Proposition 1. *A is symmetric if* $\det B_1 = \det B_0$ *for all* $u, v \in D(A)$.

Remark. Note that we must have $\gamma_3 = \delta_3 = 0$ or else the space X is not well-defined. More specifically, by taking linear combinations at either $x = 0$ or $x = 1$ we could get a new Wentzell boundary condition at that point, with a different weight and hence, a different space. We want to work in a fixed space where the weights w_0 and w_1 are uniquely defined by the problem. This is the reason for choosing $\gamma_3 = \delta_3 = 0$ and $\alpha_3 < 0 < \beta_3$.

Theorem 2. *A is symmetric if* $(\gamma_1, \gamma_2) \neq (0,0)$, $(\delta_1, \delta_2) \neq (0,0)$,

$$0 = \det \begin{bmatrix} \alpha_3 & \alpha_0 \\ \gamma_3 & \gamma_0 \end{bmatrix} - \det \begin{bmatrix} \alpha_2 & \alpha_1 \\ \gamma_2 & \gamma_1 \end{bmatrix} \tag{11}$$

and

$$0 = \det \begin{bmatrix} \beta_3 & \beta_0 \\ \delta_3 & \delta_0 \end{bmatrix} - \det \begin{bmatrix} \beta_2 & \beta_1 \\ \delta_2 & \delta_1 \end{bmatrix}. \tag{12}$$

Proof. We show that all possible boundary conditions and Proposition 1, when combined, give (11) and (12).

Case 1: $\gamma_2 \neq 0$, $\delta_2 \neq 0$. Then we have $u_0^2 = -\frac{\gamma_1}{\gamma_2}u_0^1 - \frac{\gamma_0}{\gamma_2}u_0^0$ and $u_1^2 = -\frac{\delta_1}{\delta_2}u_1^1 - \frac{\delta_0}{\delta_2}u_1^0$. In this case we see that

$$\det B_1 = \det \begin{bmatrix} 1 & -\beta_2 w_1 & \beta_1 w_1 \\ u_1^0 & u_1^1 & -\frac{\delta_1}{\delta_2}u_1^1 - \frac{\delta_0}{\delta_2}u_1^0 \\ v_1^0 & v_1^1 & -\frac{\delta_1}{\delta_2}v_1^1 - \frac{\delta_0}{\delta_2}v_1^0 \end{bmatrix} = \left[u_1^1 v_1^0 - u_1^0 v_1^1\right]\left[\frac{\delta_1\beta_2 - \delta_2\beta_1 - \delta_0\beta_3}{\delta_2\beta_3}\right]. \tag{13}$$

Similarly,

$$\det B_0 = \begin{bmatrix} 1 & \alpha_2 w_0 & -\alpha_1 w_0 \\ u_0^0 & u_0^1 & -\frac{\gamma_1}{\gamma_2}u_0^1 - \frac{\gamma_0}{\gamma_2}u_0^0 \\ v_0^0 & v_0^1 & -\frac{\gamma_1}{\gamma_2}v_0^1 - \frac{\gamma_0}{\gamma_2}v_0^0 \end{bmatrix} = \left[u_0^1 v_0^0 - u_0^0 v_0^1\right]\left[\frac{\alpha_2\gamma_1 - \alpha_1\gamma_2 - \alpha_3\gamma_0}{\alpha_3\gamma_2}\right]. \tag{14}$$

Since the boundary conditions are separated, $u_0^1 v_0^0 - u_0^0 v_0^1$ can in principle be any number. Thus, the only way $\det B_1 = \det B_0$ for all $u, v \in D(A)$ is if

$$\beta_2\delta_1 - \beta_1\delta_2 - \beta_3\delta_0 = 0$$

and

$$\alpha_2\gamma_1 - \alpha_1\gamma_2 - \alpha_3\gamma_0 = 0,$$

which reduces to (11) and (12).

Case 2: $\delta_2 = 0$, $\gamma_2 = 0$, $\delta_1 \neq 0$ and $\gamma_1 \neq 0$. Then we have $u_0^1 = -\frac{\gamma_0}{\gamma_1}u_0^0$ and $u_1^1 = -\frac{\delta_0}{\delta_1}u_1^0$. Similarly $v_0^1 = \frac{\gamma_0}{\gamma_1}v_0^0$ and $v_1^1 = -\frac{\delta_0}{\delta_1}v_1^0$.

In this case,

$$\det B_1 = \det \begin{bmatrix} 1 & -\beta_2 w_1 & \beta_1 w_1 \\ u_1^0 & -\frac{\delta_0}{\delta_1}u_1^0 & u_1^2 \\ v_1^0 & -\frac{\delta_0}{\delta_1}v_1^0 & v_1^2 \end{bmatrix} = \left[\beta_2 w_1 - \frac{\delta_0}{\delta_1}\right]\left[u_1^0 v_1^2 - u_1^2 v_1^0\right].$$

Also,

$$\det B_0 = \begin{bmatrix} 1 & \alpha_2 w_0 & -\alpha_1 w_0 \\ u_0^0 & -\frac{\gamma_0}{\gamma_1}u_0^0 & u_0^2 \\ v_0^0 & -\frac{\gamma_0}{\gamma_1}v_0^0 & v_0^2 \end{bmatrix} = \left[-\frac{\gamma_0}{\gamma_1} - \alpha_2 w_0\right]\left[u_0^0 v_0^2 - u_0^2 v_0^0\right].$$

Again using the fact that the boundary conditions are separated, $\det B_1 = \det B_0$ can hold for all $u, v \in D(A)$ if $\beta_2 w_1 - \frac{\delta_0}{\delta_1} = 0$ and $-\frac{\gamma_0}{\gamma_1} - \alpha_2 w_0 = 0$; that is,

$$0=\frac{\delta_1\beta_2-\delta_0\beta_3}{\beta_3\delta_1},\qquad 0=\frac{\alpha_2\gamma_1-\alpha_3\gamma_0}{\alpha_3\gamma_0},$$

which hold if

$$\delta_1\beta_2-\delta_0\beta_3=0$$

and

$$\alpha_2\gamma_1-\alpha_3\gamma_0=0.$$

These are (11) and (12) since $\gamma_3=\delta_3=0$ always and $\gamma_2=\delta_2=0$ in this case.

Case 3: $\delta_2=\delta_1=0$, $\delta_0\neq 0$, $\gamma_2=\gamma_1=0$, $\gamma_0\neq 0$. This is the case of Dirichlet boundary conditions at $x=0,1$. Then

$$\det B_1=\begin{bmatrix}1 & -\beta_2 w_1 & \beta_1 w_1\\ 0 & u_1^1 & u_1^2\\ 0 & v_1^1 & v_1^2\end{bmatrix}=\left[u_1^1v_1^2-v_1^1u_1^2\right].$$

Similarly,

$$\det B_0=\begin{bmatrix}1 & \alpha_2 w_0 & -\alpha_1 w_0\\ 0 & u_0^1 & u_0^2\\ 0 & v_0^1 & v_0^2\end{bmatrix}=\left[u_0^1v_0^2-u_0^2v_0^1\right].$$

Thus, in this case, the fact that the boundary conditions are separated shows that $\det B_1\neq\det B_0$; hence A cannot be symmetric with these boundary conditions. Indeed, for example suppose at $x=0$, we have $\det\left[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \gamma_2 & \gamma_1\end{smallmatrix}\right]=0$ and A is symmetric. Then it follows that $\det\left[\begin{smallmatrix}\alpha_3 & \alpha_0\\ \gamma_3 & \gamma_0\end{smallmatrix}\right]=0$. But $\gamma_3=0$, so $\gamma_0\alpha_3=0$. This contradicts the facts that $w_0=-\frac{1}{\alpha_3}>0$ and $\gamma_0\neq 0$. Hence, A is not symmetric.

Since the boundary conditions are separated, all other possible combinations follow from portions of the previous calculations. This completes the proof of Theorem 2. □

Corollary 3. *The operator A with Wentzell and Dirichlet boundary conditions at one endpoint is not symmetric.*

4. Quasiaccretivity of A

In this section we consider the question of quasiaccretivity. More specifically when is it true that

$$\operatorname{Re}\langle Au,u\rangle\geqslant\eta|||u|||^2 \tag{15}$$

for some $\eta\in\mathbb{R}$ and all $u\in D(A)$?

We begin this section with an example, but first we note that for $u\in D(A)$,

$$\begin{aligned}\langle Au,u\rangle&=\int_0^1\left|u''(x)\right|^2dx+u'''(x)\overline{u(x)}\Big|_0^1-u''(x)\overline{u'(x)}\Big|_0^1+\sum_{i=0}^{1}Au(i)\overline{u(i)}w_i\\&=\|u''\|_2^2+\left[-\alpha_2u''(0)-\alpha_1u'(0)-\alpha_0u(0)\right]\overline{u(0)}w_0\\&\quad+\left[-\beta_2u''(1)-\beta_1u'(1)-\beta_0u(1)\right]\overline{u(1)}w_1-u''(1)\overline{u'(1)}+u''(0)\overline{u'(0)}.\end{aligned}\tag{16}$$

Example. Here we show that the operator A with one general Wentzell and one Robin boundary condition at each endpoint need not be quasiaccretive in the nonsymmetric case. We consider the boundary conditions

$$u''''(0) - u'''(0) + u''(0) + u'(0) + u(0) = 0,$$
$$u'(0) + u(0) = 0,$$
$$u''''(1) + u'''(1) + u''(1) + u'(1) + u(1) = 0,$$
$$u'(1) + u(1) = 0.$$

Clearly, $\alpha_3 = -1$, $\alpha_2 = \alpha_1 = \alpha_0 = 1$, $\gamma_3 = \gamma_2 = 0$, $\gamma_1 = \gamma_0 = 1$, $\beta_3 = \beta_2 = \beta_1 = \beta_0 = 1$, $\delta_3 = \delta_2 = 0$, and $\delta_1 = \delta_0 = 1$.

Note in these boundary conditions (11) is not satisfied, but (12) holds. Let

$$v_n(x) = -\cos nx + n \sin nx + 5\cos x - n^2 \sin x - 3 \sin 2x + 2\cos 2x.$$

Let ζ_n be an infinitely differentiable function on $[0, 1]$ with $\operatorname{supp}(\zeta_n) \subseteq [0, k_n)$, $0 < k_n < 1$, such that $\zeta_n(x) = 1$ on $[0, \epsilon_n)$ and $\zeta_n(x) = 0$ on $(k_n, 1]$. Set $u_n = v_n \zeta_n$. Clearly, since v_n satisfies the boundary conditions at $x = 0$, so does u_n. Also, since ζ_n vanishes on $(k_n, 1]$, u_n vanishes on a neighborhood of 1, so u_n satisfies the boundary conditions at $x = 1$. Notice also that

$$-u_n''(0)\overline{u_n(0)} = -6n^2 + 78,$$

and $\langle u_n, u_n\rangle = |||u_n|||^2$ is clearly bounded by choosing $\epsilon_n \approx \frac{1}{n^2}$ and $k_n \approx \frac{2}{n^2}$. Hence, using (16), we see that $\langle Au_n, u_n\rangle \to -\infty$ as $n \to \infty$.

For any nonsymmetric general Wentzell and Robin boundary conditions, we suspect that quasiaccretively fails, but we do not have a proof in general.

In the following theorem we address the issue of quasiaccretivity without the assumption of symmetry.

Theorem 4. *The operator A on $D(A)$ with boundary conditions* (2)–(5)*, $\alpha_3 < 0 < \beta_3$ and $\delta_3 = \gamma_3 = 0$, is quasiaccretive if* one *of the following conditions holds at $x = 0$:*

$$\gamma_1 \neq 0, \qquad \gamma_2 = 0 \quad \text{and} \quad \gamma_0 = \frac{\alpha_2\gamma_1}{\alpha_3}, \tag{17}$$

$$\gamma_2 \neq 0, \qquad \frac{\gamma_1}{\gamma_2} < 0, \tag{18}$$

$$\gamma_2 \neq 0, \qquad \gamma_1 = 0 \quad \text{and} \quad \gamma_0 = \frac{\alpha_1\gamma_2}{\alpha_3} \tag{19}$$

and one *of the following conditions holds at $x = 1$:*

$$\delta_1 \neq 0, \qquad \delta_2 = 0, \qquad \delta_0 = \frac{\beta_2\delta_1}{\beta_3}, \tag{20}$$

$$\delta_2 \neq 0, \qquad \frac{\delta_1}{\delta_2} > 0, \tag{21}$$

$$\delta_2 \neq 0, \qquad \delta_1 = 0 \quad \text{and} \quad \delta_0 = \frac{\beta_1\delta_2}{\beta_3}. \tag{22}$$

Note that the conditions in (17) and (20) are the conditions for symmetry of A at each endpoint respectively by Theorem 2.

Combining Theorems 2 and 4, we obtain the following theorem.

Theorem 5. *The operator A on $D(A)$ with boundary conditions* (2)–(5), *$\alpha_3 < 0 < \beta_3$ and $\gamma_3 = \delta_3 = 0$ is symmetric and semibounded if* one *of the following boundary conditions holds at $x = 0$*:

$$\gamma_1 \neq 0, \qquad \gamma_2 = 0, \qquad \gamma_0 = \frac{\alpha_2\gamma_1}{\alpha_3}, \tag{23}$$

$$\gamma_2 \neq 0, \qquad \frac{\gamma_1}{\gamma_2} < 0, \qquad \gamma_0 = \frac{\alpha_2\gamma_1 - \alpha_1\gamma_2}{\alpha_3}, \tag{24}$$

$$\gamma_2 \neq 0, \qquad \gamma_1 = 0 \quad \textit{and} \quad \gamma_0 = 0, \quad \textit{and } \alpha_1 = 0, \tag{25}$$

and one *of the following conditions holds at $x = 1$*:

$$\delta_1 \neq 0, \qquad \delta_2 = 0, \qquad \delta_0 = \frac{\beta_2\delta_1}{\beta_3}, \tag{26}$$

$$\delta_2 \neq 0, \qquad \frac{\delta_1}{\delta_2} > 0, \qquad \delta_0 = \frac{\beta_2\delta_1 - \beta_1\delta_2}{\beta_3}, \tag{27}$$

$$\delta_2 \neq 0, \qquad \delta_1 = 0 \quad \textit{and} \quad \delta_0 = 0, \quad \textit{and } \beta_1 = 0. \tag{28}$$

Proof of Theorem 4. *Case* 1: $\gamma_2 = 0$, $\gamma_1 \neq 0$ and $\delta_2 = 0$, $\delta_1 \neq 0$. In this case we have $u_0^1 = -\frac{\gamma_0}{\gamma_1}u_0^0$ and $u_1^1 = -\frac{\delta_0}{\delta_1}u_1^0$. This is the case of Neumann or Robin boundary conditions at both ends. Then, using (16),

$$\begin{aligned}\langle Au, u\rangle = {}& \|u''\|_2^2 + \frac{\alpha_2}{\alpha_3}u_0^2\overline{u_0^0} - \frac{\alpha_1\gamma_0}{\alpha_3\gamma_1}\left|u_0^0\right|^2 + \frac{\alpha_0}{\alpha_3}\left|u_0^0\right|^2 - \frac{\beta_2}{\beta_3}u_1^2\overline{u_1^0} + \frac{\beta_1\delta_0}{\beta_3\delta_1}\left|u_1^0\right|^2 - \frac{\beta_0}{\beta_3}\left|u_1^0\right|^2 \\ &+ \left[\frac{\delta_0}{\delta_1}u_1^2\overline{u_1^0} + \frac{\gamma_0}{\gamma_1}u_0^2\overline{u_0^0}\right]\end{aligned} \tag{29}$$

and so

$$\begin{aligned}\operatorname{Re}\langle Au, u\rangle = {}& \|u''\|_2^2 + \frac{\alpha_0\gamma_1 - \alpha_1\gamma_0}{\alpha_3\gamma_1}\left|u_0^0\right|^2 + \frac{\beta_1\delta_0 - \beta_0\delta_1}{\beta_3\delta_1}\left|u_1^0\right|^2 + \frac{\alpha_2\gamma_1 - \alpha_3\gamma_0}{\alpha_3\gamma_1}\operatorname{Re}\bigl(u_0^2\overline{u_0^0}\bigr) \\ &+ \frac{\beta_3\delta_0 - \beta_2\delta_1}{\beta_3\delta_1}\operatorname{Re}\bigl(u_1^2\overline{u_1^0}\bigr).\end{aligned} \tag{30}$$

Let

$$\widetilde{a_0} = \frac{\alpha_0\gamma_1 - \alpha_1\gamma_0}{\alpha_3\gamma_1}, \qquad \widetilde{a_1} = \frac{\beta_1\delta_0 - \beta_0\delta_1}{\beta_3\delta_1},$$
$$\widetilde{b_0} = \frac{\alpha_2\gamma_1 - \alpha_3\gamma_0}{\alpha_3\gamma_1}, \qquad \widetilde{b_1} = \frac{\beta_3\delta_0 - \beta_2\delta_1}{\beta_3\delta_1}.$$

Using (17) and (20) we see that $\widetilde{b_0} = \widetilde{b_1} = 0$, and so

$$\begin{aligned}\operatorname{Re}\langle Au, u\rangle &= \|u''\|_2^2 + \widetilde{a_0}\left|u_0^0\right|^2 + \widetilde{a_1}\left|u_1^0\right|^2 \\ &= \|u''\|_2^2 + \frac{\alpha_1\gamma_0 - \alpha_0\gamma_1}{\gamma_1}\left|u_0^0\right|^2 w_0 + \frac{\beta_1\delta_0 - \beta_0\delta_1}{\delta_1}\left|u_1^0\right|^2 w_1 \\ &\geqslant \widetilde{\eta_0}|||u|||^2,\end{aligned}$$

where $\widetilde{\eta_0} = \min\{0, \frac{\alpha_1\gamma_0 - \alpha_0\gamma_1}{\gamma_1}, \frac{\beta_1\delta_0 - \beta_0\delta_1}{\delta_1}\}$.

Case 2: $\gamma_2 \neq 0$, $\delta_2 \neq 0$. Then we have $u_0^2 = -\frac{\gamma_1}{\gamma_2} u_0^1 - \frac{\gamma_0}{\gamma_2} u_0^0$ and $u_1^2 = -\frac{\delta_1}{\delta_2} u_1^1 - \frac{\delta_0}{\delta_2} u_1^0$. Then (16) becomes

$$\begin{aligned}\langle Au, u\rangle = {}& \|u''\|_2^2 - \frac{\alpha_2\gamma_1}{\alpha_3\gamma_2} u_0^1 \overline{u_0^0} - \frac{\alpha_2\gamma_0}{\alpha_3\gamma_2} \left|u_0^0\right|^2 + \frac{\alpha_1}{\alpha_3} u_0^1 \overline{u_0^0} + \frac{\alpha_0}{\alpha_3} \left|u_0^0\right|^2 + \frac{\beta_2\delta_1}{\beta_3\delta_2} u_1^1 \overline{u_1^0} \\ & + \frac{\beta_2\delta_0}{\beta_3\delta_2} \left|u_1^0\right|^2 - \frac{\beta_1}{\beta_3} u_1^1 \overline{u_1^0} - \frac{\beta_0}{\beta_3} \left|u_1^0\right|^2 + \frac{\delta_1}{\delta_2} \left|u_1^1\right|^2 + \frac{\delta_0}{\delta_2} u_1^0 \overline{u_1^1} - \frac{\gamma_1}{\gamma_2} \left|u_0^1\right|^2 - \frac{\gamma_0}{\gamma_2} u_0^0 \overline{u_0^1}.\end{aligned}$$

It follows that

$$\begin{aligned}\operatorname{Re}\langle Au, u\rangle = {}& \|u''\|_2^2 + \left(\frac{\alpha_1}{\alpha_3} - \frac{\alpha_2\gamma_1}{\alpha_3\gamma_2} - \frac{\gamma_0}{\gamma_2}\right) \operatorname{Re}\left(u_0^1 \overline{u_0^0}\right) - \frac{\gamma_1}{\gamma_2} \left|u_0^1\right|^2 + \left(\frac{\alpha_0}{\alpha_3} - \frac{\alpha_2\gamma_0}{\alpha_3\gamma_2}\right) \left|u_0^0\right|^2 \\ & + \left(\frac{\beta_2\delta_1}{\beta_3\delta_2} - \frac{\beta_1}{\beta_3} + \frac{\delta_0}{\delta_2}\right) \operatorname{Re}\left(u_1^1 \overline{u_1^0}\right) + \frac{\delta_1}{\delta_2} \left|u_1^1\right|^2 + \left(\frac{\beta_2\delta_0}{\beta_3\delta_2} - \frac{\beta_0}{\beta_3}\right) \left|u_1^0\right|^2. \qquad (31)\end{aligned}$$

Set

$$\begin{aligned} & a_0 = -\frac{\gamma_1}{\gamma_2}, \qquad a_1 = \frac{\delta_1}{\delta_2}, \\ & b_0 = \frac{\alpha_1\gamma_2 - \alpha_2\gamma_1 - \alpha_3\gamma_0}{\alpha_3\gamma_2}, \qquad b_1 = \frac{\beta_2\delta_1 - \beta_1\delta_2 + \beta_3\delta_0}{\beta_3\delta_2}, \\ & c_0 = \frac{\alpha_2\gamma_0 - \alpha_0\gamma_2}{\gamma_2}, \qquad c_1 = \frac{\beta_2\delta_0 - \beta_0\delta_2}{\delta_2}. \end{aligned}$$

Then by (18) or (19) and (21) or (22), $a_0 \geqslant 0$ and $a_1 \geqslant 0$. Then (31) is

$$\begin{aligned}\operatorname{Re}\langle Au, u\rangle = {}& \|u''\|_2^2 + a_0 \left|u_0^1\right|^2 + b_0 \operatorname{Re}\left(u_0^1 \overline{u_0^0}\right) + c_0 \left|u_0^0\right|^2 w_0 \\ & + a_1 \left|u_1^1\right|^2 + b_1 \operatorname{Re}\left(u_1^1 \overline{u_1^0}\right) + c_1 \left|u_1^0\right|^2 w_1. \qquad (32)\end{aligned}$$

If $a_0, a_1 > 0$, we can write

$$\begin{aligned}\operatorname{Re}\langle Au, u\rangle = {}& \|u''\|_2^2 + a_0 \left(\left|u_0^1\right|^2 + \frac{b_0}{a_0} \operatorname{Re}\left(u_0^1 \overline{u_0^0}\right) + \frac{b_0^2}{4a_0^2} \left|u_0^0\right|^2\right) + \sum_{i=0}^{1} c_i \left|u_i^0\right|^2 w_i \\ & - \frac{b_0^2}{4a_0} \left|u_0^0\right|^2 - \frac{b_1^2}{4a_1} \left|u_1^0\right|^2 + a_1 \left(\left|u_1^1\right|^2 + \frac{b_1}{a_1} \operatorname{Re}\left(u_1^1 \overline{u_1^0}\right) + \frac{b_1^2}{4a_1^2} \left|u_1^0\right|^2\right) \\ = {}& \|u''\|_2^2 + a_0 \left|u_0^1 + \frac{b_0}{2a_0} u_0^0\right|^2 + a_1 \left|u_1^1 + \frac{b_1}{2a_1} u_1^0\right|^2 \\ & + \left(c_0 + \frac{b_0^2\alpha_3}{4a_0}\right) \left|u_0^0\right|^2 w_0 + \left(c_1 - \frac{b_1^2\beta_3}{4a_1}\right) \left|u_1^0\right|^2 w_1 \\ \geqslant {}& \left(c_0 + \frac{b_0^2\alpha_3}{4a_0}\right) \left|u_0^0\right|^2 w_0 + \left(c_1 - \frac{b_1^2\beta_3}{4a_1}\right) \left|u_1^0\right|^2 w_1 \\ \geqslant {}& \eta_1 \|\!|u|\!\|^2,\end{aligned}$$

where $\eta_1 = \min\{0, c_0 + \frac{b_0^2\alpha_3}{4a_0}, c_1 - \frac{b_1^2\beta_3}{4a_1}\}$.

Thus, the operator A is always quasiaccretive if $a_0, a_1 > 0$, that is, if (18) and (21) hold (even if A is not symmetric). Careful analysis of (32), shows A is quasiaccretive if one of the following conditions holds:

$$a_0 > 0, \qquad a_1 > 0 \quad \big(\text{which corresponds to (18) and (21)}\big), \tag{33}$$
$$a_0 > 0, \qquad a_1 = 0, \qquad b_1 = 0 \quad \big(\text{which corresponds to (18) and (22)}\big), \tag{34}$$
$$a_0 = 0, \qquad a_1 > 0, \qquad b_0 = 0 \quad \big(\text{which corresponds to (19) and (21)}\big), \tag{35}$$
$$a_0 = b_0 = 0, \qquad a_1 = b_1 = 0 \quad \big(\text{which corresponds to (19) and (22)}\big). \tag{36}$$

This completes the proof of Theorem 4. □

Proof of Theorem 5. Condition (33) gives quasiaccretivity even if A is not symmetric. Notice that $a_1 = b_1 = 0$ together with the condition (12) for symmetry of A at $x = 1$ holds if and only if $\delta_0 = 0$. Condition (35) requires $a_0 = b_0 = 0$, that is, $\alpha_1\gamma_2 - \alpha_2\gamma_1 = \alpha_3\gamma_0$. If, in this case, A is also symmetric, then combining with condition (11) yields $\gamma_0 = 0$. But $a_0 = 0$ (respectively $b_0 = 0$) implies $\delta_1 = 0$ (respectively $\alpha_1 = \frac{\alpha_3\gamma_0}{\gamma_2}$). Hence, $\alpha_1 = 0$ in this case. Now Theorem 5 follows. □

5. The range condition

In this section we assume that A is quasiaccretive. We must solve the equation

$$\lambda u + Au = h \tag{37}$$

in [0, 1] for some $\lambda \in \mathbb{R}$ with λ sufficiently large and for each $h \in C[0, 1]$. Using (2), (3) we see

$$\alpha_3 u'''(0) + \alpha_2 u''(0) + \alpha_1 u'(0) + (\alpha_0 - \lambda)u(0) = -h(0), \tag{38}$$
$$\beta_3 u'''(1) + \beta_2 u''(1) + \beta_1 u'(1) + (\beta_0 - \lambda)u(1) = -h(1). \tag{39}$$

Multiplying (37) by $\bar{v} \in D(A)$ and integrating over (0, 1) yields

$$\lambda \int_0^1 u\bar{v}\,dx + \int_0^1 (Au)\bar{v}\,dx = \int_0^1 h\bar{v}\,dx,$$
$$\lambda \int_0^1 u\bar{v}\,dx - \int_0^1 u'''\bar{v}'\,dx + u'''\bar{v}\big|_0^1 = \int_0^1 h\bar{v}\,dx.$$

Using the boundary conditions (38), (39) yields

$$\lambda \int_0^1 u\bar{v}\,dx + \left[\frac{\alpha_2 u''(0) + \alpha_1 u'(0) + (\alpha_0 - \lambda)u(0)}{\alpha_3}\right]\overline{v(0)}$$
$$- \left[\frac{\beta_2 u''(1) + \beta_1 u'(1) + (\beta_0 - \lambda)u(1)}{\beta_3}\right]\overline{v(1)} - \int_0^1 u'''\bar{v}'\,dx$$
$$= \int_0^1 h\bar{v}\,dx + \sum_{i=0}^{1} h(i)\overline{v(i)}w_i.$$

Another integration by parts yields

$$\lambda \int_0^1 u\bar{v}\,dx - \big[\alpha_2 u''(0) + \alpha_1 u'(0) + (\alpha_0 - \lambda)u(0)\big]\overline{v(0)}w_0$$
$$- \big[\beta_2 u''(1) + \beta_1 u'(1) + (\beta_0 - \lambda)u(1)\big]\overline{v(1)}w_1 + \int_0^1 u''\bar{v}''\,dx - u''\bar{v}'\big|_0^1$$
$$= \int_0^1 h\bar{v}\,dx + \sum_{i=0}^1 h(i)\overline{v(i)}w_i. \tag{40}$$

Case 1: $\gamma_2 = \delta_2 = 0$, $\gamma_1 \neq 0$, $\delta_1 \neq 0$ so that $u_0^1 = -\frac{\gamma_0}{\gamma_1}u_0^0$ and $u_1^1 = -\frac{\delta_0}{\delta_1}u_1^0$. This is the case of (17) and (20) in Theorem 4. In this case (40) becomes, for $u, v \in D(A)$,

$$\lambda \int_0^1 u\bar{v}\,dx + \lambda \sum_{i=0}^1 u(i)\overline{v(i)}w_i - \big[\alpha_2 u''(0) + \alpha_1 u'(0) + \alpha_0 u(0)\big]\overline{v(0)}w_0$$
$$- \big[\beta_2 u''(1) + \beta_1 u'(1) + \beta_0 u(1)\big]\overline{v(1)}w_1 + \int_0^1 u''\bar{v}''\,dx + \frac{\delta_0}{\delta_1}u''(1)\overline{v(1)}$$
$$- \frac{\gamma_0}{\gamma_1}u''(0)\overline{v(0)}$$
$$= \int_0^1 h\bar{v}\,dx + \sum_{i=0}^1 h(i)\overline{v(i)}w_i. \tag{41}$$

Recall that (17) and (20) say that $\frac{\beta_2}{\beta_3} = \frac{\delta_0}{\delta_1}$ and $\frac{\alpha_2}{\alpha_3} = \frac{\gamma_0}{\gamma_1}$, so (41) reduces to

$$\lambda \int_0^1 u\bar{v}\,dx + \lambda \sum_{i=0}^1 u(i)\overline{v(i)}w_i + \left[\frac{\alpha_1\gamma_0 - \alpha_0\gamma_1}{\gamma_1}\right]u(0)\overline{v(0)}w_0$$
$$+ \left[\frac{\beta_1\delta_0 - \beta_0\delta_1}{\delta_1}\right]u(1)\overline{v(1)}w_1 + \int_0^1 u''\bar{v}''\,dx$$
$$= \int_0^1 h\bar{v}\,dx + \sum_{i=0}^1 h(i)\overline{v(i)}w_i. \tag{42}$$

Let $L(u,v)$ be the left-hand side of (42), and let $F(v)$ be the right-hand side. Let $\mathcal{K}$ be the completion of $C^2[0,1]$ in the norm

$$\|u\|_{\mathcal{K}} = \big(|||u|||^2 + \|u''\|^2_{L^2[0,1]}\big)^{\frac{1}{2}}. \tag{43}$$

Then

$$\big|L(u,v)\big| \leqslant \max\big\{|\lambda|, 1\big\}\|u\|_{\mathcal{K}}\|v\|_{\mathcal{K}} + \max\left\{\left|\frac{\alpha_1\gamma_0 - \alpha_0\gamma_1}{\gamma_1}\right|, \left|\frac{\beta_1\delta_0 - \beta_0\delta_1}{\delta_1}\right|\right\}\|u\|_{\mathcal{K}}\|v\|_{\mathcal{K}}.$$

If $h \in \mathcal{K}$, then

$$\left|F(v)\right| \leqslant \|h\|_{\mathcal{K}}\|v\|_{\mathcal{K}}.$$

We also note that

$$\operatorname{Re} L(u,u) \geqslant \min\left\{\lambda, \lambda + \left(\frac{\alpha_1\gamma_0 - \alpha_0\gamma_1}{\gamma_1}\right), \lambda + \left(\frac{\beta_1\delta_0 - \beta_0\delta_1}{\delta_1}\right), 1\right\}\|u\|_{\mathcal{K}}^2.$$

Choose $\lambda_0 > \max\{1, (\frac{\alpha_1\gamma_0-\alpha_0\gamma_1}{\gamma_1}), (\frac{\beta_1\delta_0-\beta_0\delta_1}{\delta_1})\}$. It follows that if $\lambda > \lambda_0$,

$$\operatorname{Re} L(u,u) \geqslant \epsilon\|u\|_{\mathcal{K}}^2$$

for some $\epsilon > 0$. Hence we may apply the Lax–Milgram Lemma to obtain a unique $u \in \mathcal{K}$ which satisfies

$$L(u,v) = F(v) \quad \text{for all } v \in \mathcal{K}.$$

This is our weak solution of (37) when A is quasidissipative and satisfies the boundary conditions (2)–(5). We again point out that for the case of the boundary conditions (17) and (20), A is quasidissipative exactly when A is symmetric.

Case 2: $\gamma_2 \neq 0$, $\delta_2 \neq 0$. In this case our boundary conditions take the form $u_0^2 = -\frac{\gamma_1}{\gamma_2}u_0^1 - \frac{\gamma_0}{\gamma_2}u_0^0$ and $u_1^2 = -\frac{\delta_1}{\delta_2}u_1^1 - \frac{\delta_0}{\delta_2}u_1^0$. In this case (40) becomes

$$\begin{aligned}
&\lambda\int_0^1 u\bar{v}\,dx + \lambda\sum_{i=0}^1 u(i)\overline{v(i)}w_i + \left[\frac{\alpha_2\gamma_1 - \alpha_1\gamma_2}{\gamma_2}\right]u'(0)\overline{v(0)}w_0 \\
&\quad + \left[\frac{\alpha_2\gamma_0 - \alpha_0\gamma_2}{\gamma_2}\right]u(0)\overline{v(0)}w_0 + \left[\frac{\beta_2\delta_1 - \beta_1\delta_2}{\delta_2}\right]u'(1)\overline{v(1)}w_1 \\
&\quad + \left[\frac{\beta_2\delta_0 - \beta_0\delta_2}{\delta_2}\right]u(1)\overline{v(1)}w_1 + \int_0^1 u''\bar{v}''\,dx + \frac{\delta_1}{\delta_2}u'(1)\overline{v'(1)} + \frac{\delta_0}{\delta_2}u(1)\overline{v'(1)} \\
&\quad - \frac{\gamma_1}{\gamma_2}u'(0)\overline{v'(0)} - \frac{\gamma_0}{\gamma_2}u(0)\overline{v'(0)} \\
&= \int_0^1 h\bar{v}\,dx + \sum_{i=0}^1 h(i)\overline{v(i)}w_i.
\end{aligned} \tag{44}$$

Let $\mathcal{K}$ be the completion of $C^2[0,1]$ in the norm given by (43).

Let $L_1(u,v)$ be the left-hand side of (44), and let $F_1(v)$ be the right-hand side. Let us first consider the case where A is symmetric, $\frac{\delta_1}{\delta_2} > 0$, and $\frac{\gamma_1}{\gamma_2} < 0$. Also, $\alpha_2\gamma_1 - \alpha_1\gamma_2 = \alpha_3\gamma_0$ and $\beta_3\delta_0 = \beta_2\delta_1 - \beta_1\delta_2$. Then using the notation of Section 4,

$$\lambda\int_0^1 u\bar{v}\,dx + \lambda\sum_{i=0}^1 u(i)\overline{v(i)}w_i - \frac{\gamma_0}{\gamma_2}u'(0)\overline{v(0)} + c_0u(0)\overline{v(0)}w_0 + \frac{\delta_0}{\delta_2}u'(1)\overline{v(1)}$$

$$
\begin{aligned}
&+c_1u(1)\overline{v(1)}w_1+\frac{\delta_1}{\delta_2}u'(1)\overline{v'(1)}+\frac{\delta_0}{\delta_2}u(1)\overline{v'(1)}-\frac{\gamma_1}{\gamma_2}u'(0)\overline{v'(0)}\\
&-\frac{\gamma_0}{\gamma_2}u(0)\overline{v'(0)}+\int_0^1 u''\bar{v}''\,dx\\
={}&\int_0^1 h\bar{v}\,dx+\sum_{i=0}^{1}h(i)\overline{v(i)}w_i. \qquad (45)
\end{aligned}
$$

Clearly,

$$\left|F_1(v)\right| \leqslant \|h\|_{\mathcal{K}}\|v\|_{\mathcal{K}}.$$

We need the following lemma.

Lemma 6. *Let $u \in W^{2,2}[0,1]$. Then for $0 \leqslant x^* \leqslant 1$*

$$\left|u'\left(x^*\right)\right| \leqslant \sqrt{12}\left(\|u\|_2+\|u''\|_2\right).$$

Proof. Fix $x^* \in [0,1]$. Clearly, for $0 \leqslant y \leqslant x^* \leqslant 1$

$$u'\left(x^*\right)=u'(y)+\int_y^{x^*} u''(t)\,dt$$

and for $0 \leqslant r, s \leqslant 1$,

$$\left|(s-r)u'\left(x^*\right)\right| \leqslant \left|u(s)-u(r)\right|+\|u''\|_1 \leqslant \left|u(s)-u(r)\right|+\|u''\|_2. \qquad (46)$$

Choose s so that

$$u(s)=\text{average}(u)=\int_0^1 u(t)\,dt. \qquad (47)$$

It is well known from probability theory that the L^2-norm of a random variable minus a constant is minimized when the constant is the mean of the random variable; in particular

$$\left\|u-u(s)\right\|_2 \leqslant \|u\|_2 \qquad (48)$$

since s is chosen as in (47). Taking the L^2 norm on both sides of (46), we see

$$\left\|(r-s)u'\left(x^*\right)\right\|_2 \leqslant \|u\|_2+\|u''\|_2.$$

Notice that

$$\left\|(r-s)u'\left(x^*\right)\right\|_2=\left|u'\left(x^*\right)\right|\left(\int_0^1 (r-s)^2\,dr\right)^{\frac{1}{2}}.$$

Since s is fixed by (48), an elementary calculation shows that

$$\left\|(r-s)u'\left(x^*\right)\right\|_2=\left|u'\left(x^*\right)\right|\left(\frac{1}{3}-s+s^2\right)^{\frac{1}{2}} \geqslant \frac{1}{\sqrt{12}}\left|u'\left(x^*\right)\right|.$$

Thus,

$$\left|u'\left(x^*\right)\right| \leqslant \sqrt{12}\left(\|u\|_2 + \|u''\|_2\right).$$

This completes the proof of the lemma. □

Now

$$\begin{aligned}
\left|L_1(u,v)\right| &\leqslant |\lambda|\,|||u|||\,|||v||| + \|u''\|_{L^2}\|v''\|_{L^2} + \left|\frac{\gamma_0}{\gamma_2}\right|\left(\left|u'(0)\right|\left|v(0)\right| + \left|u(0)\right|\left|v'(0)\right|\right)\\
&\quad + \sum_{i=0}^{1}|c_i|\left|u(i)\right|\left|v(i)\right|w_i + \left|\frac{\delta_0}{\delta_2}\right|\left(\left|u'(1)\right|\left|v(1)\right| + \left|u(1)\right|\left|v'(1)\right|\right)\\
&\quad + \frac{\delta_1}{\delta_2}\left|u'(1)\right|\left|v'(1)\right| - \frac{\gamma_1}{\gamma_2}\left|u'(0)\right|\left|v'(0)\right|\\
&\leqslant \max\left\{1, |\lambda|, |c_0|, |c_1|, 12\left|\frac{\delta_0\beta_3}{\delta_2}\right|, 12\left|\frac{\gamma_0\alpha_3}{\gamma_2}\right|, \frac{144\delta_1}{\delta_2}, 144\left|\frac{\gamma_1}{\gamma_2}\right|\right\}\|u\|_{\mathcal{K}}\|v\|_{\mathcal{K}}
\end{aligned}$$

by Lemma 6. Furthermore, we see

$$\begin{aligned}
\operatorname{Re} L_1(u,u) &= \lambda|||u|||^2 - \frac{2\gamma_0}{\gamma_2}\operatorname{Re}\left(u'(0)\overline{u(0)}\right) + c_0\left|u(0)\right|^2 w_0 + \frac{2\delta_0}{\delta_2}\operatorname{Re}\left(u'(1)\overline{u(1)}\right)\\
&\quad + c_1\left|u(1)\right|^2 w_1 - \frac{\gamma_1}{\gamma_2}\left|u'(0)\right|^2 + \frac{\delta_1}{\delta_2}\left|u'(1)\right|^2 + \|u''\|_{L^2}^2\\
&= \lambda|||u|||^2 + \|u''\|_{L^2}^2 - \frac{\gamma_1}{\gamma_2}\left(\left|u'(0)\right|^2 + \frac{2\gamma_0}{\gamma_1}\operatorname{Re}\left(u'(0)\overline{u(0)}\right) + \frac{\gamma_0^2}{\gamma_1^2}\left|u(0)\right|^2\right)\\
&\quad + \left(c_0 - \frac{\gamma_0^2\alpha_3}{\gamma_1\gamma_2}\right)\left|u(0)\right|^2 w_0\\
&\quad + \frac{\delta_1}{\delta_2}\left(\left|u'(1)\right|^2 + \frac{2\delta_0}{\delta_1}\operatorname{Re}\left(u'(1)\overline{u(1)}\right) + \frac{\delta_0^2}{\delta_1^2}\left|u(1)\right|^2\right)\\
&\quad + \left(c_1 - \frac{\delta_0^2\beta_3}{\delta_1\delta_2}\right)\left|u(1)\right|^2 w_1\\
&= \lambda|||u|||^2 + \|u''\|_{L^2}^2 - \frac{\gamma_1}{\gamma_2}\left|u'(0) + \frac{\gamma_0}{\gamma_1}u(0)\right|^2 + \frac{\delta_1}{\delta_2}\left|u'(1) + \frac{\delta_0}{\delta_1}u(1)\right|^2\\
&\quad + \left(c_0 - \frac{\gamma_0^2\alpha_3}{\gamma_1\gamma_2}\right)\left|u(0)\right|^2 w_0 + \left(c_1 - \frac{\delta_0^2\beta_3}{\delta_1\delta_2}\right)\left|u(1)\right|^2 w_1.
\end{aligned}$$

Choose

$$\lambda_0 > 2\max\left\{1, \left(c_0 - \frac{\gamma_0^2\alpha_3}{\gamma_1\gamma_2}\right), \left(c_1 - \frac{\delta_0^2\beta_3}{\delta_1\delta_2}\right)\right\}.$$

Then if $\lambda > \lambda_0$, $\operatorname{Re} L_1(u,u) \geqslant \epsilon_1\|u\|_{\mathcal{K}}^2$ for some $\epsilon_1 > 0$; here we used $\frac{\gamma_1}{\gamma_2} < 0$ and $\frac{\delta_1}{\delta_2} > 0$.

Applying the Lax–Milgram Lemma, we obtain a unique $u \in \mathcal{K}$ which satisfies

$$L_1(u,v) = F_1(v) \quad \text{for all } v \in \mathcal{K}.$$

This u is our weak solution of (37) when A is symmetric and satisfies the boundary conditions for this case. If $\gamma_1 = 0$, by the symmetry condition we must have $\gamma_0 = 0$, and so

$$\begin{aligned}\operatorname{Re} L_1(u,u) &= \lambda |||u|||^2 - \alpha_0 |u(0)|^2 w_0 + \frac{2\delta_0}{\delta_2} \operatorname{Re}\bigl(u'(1)\overline{u(1)}\bigr) + c_1 |u(1)|^2 w_1 \\ &\quad + \frac{\delta_1}{\delta_2} |u'(1)|^2 + \|u''\|_{L^2}^2 \\ &\geqslant \lambda |||u|||^2 + \|u''\|_{L^2}^2 + \frac{\delta_1}{\delta_2} \left| u'(1) + \frac{\delta_0}{\delta_1} u(1) \right|^2 + \left(c_1 - \frac{\delta_0^2 \beta_3}{\delta_1 \delta_2} \right) |u(1)|^2 w_1 \\ &\quad - \alpha_0 |u(0)|^2 w_0.\end{aligned}$$

Choose

$$\lambda_0 > 2 \max\left\{ 1, -\alpha_0, \left(c_1 - \frac{\delta_0^2 \beta_3}{\delta_1 \delta_2} \right) \right\}.$$

Then if $\lambda > \lambda_0$, so we have $\operatorname{Re} L_1(u,u) \geqslant \epsilon_1 \|u\|_{\mathcal{K}}^2$ for some $\epsilon_1 > 0$. Hence we again obtain our weak solution via the Lax–Milgram Lemma.

Now let us consider this case without the assumption that A is symmetric. Here we have (44), and $\mathcal{K}$ is defined as before. Let $L_2(u,v)$ be the left-hand side of (44). Then

$$\begin{aligned}|L_2(u,v)| &\leqslant \lambda |||u||| \, |||v||| + \|u''\|_{L^2} \|v''\|_{L^2} + \left| \frac{\gamma_0}{\gamma_2} \right| \bigl(|u(0)| |v'(0)| \bigr) + \left| \frac{\delta_0}{\delta_2} \right| \bigl(|u(1)| |v'(1)| \bigr) \\ &\quad + \left| \frac{\gamma_1}{\gamma_2} \right| \bigl(|u'(0)| |v'(0)| \bigr) + \left| \frac{\delta_1}{\delta_2} \right| \bigl(|u'(1)| |v'(1)| \bigr) \\ &\quad + \left| \frac{\alpha_2 \gamma_1 - \alpha_1 \gamma_2}{\gamma_2 \alpha_3} \right| \bigl(|u'(0)| |v(0)| \bigr) + \left| \frac{\beta_2 \delta_1 - \beta_1 \delta_2}{\delta_2 \beta_3} \right| \bigl(|u'(1)| |v(1)| \bigr) \\ &\quad + \sum_{i=0}^{1} |c_i| |u(i)| |v(i)| w_i,\end{aligned}$$

where c_0 and c_1 were defined in the proof of Theorem 4. Thus, again using Lemma 6, we obtain

$$\begin{aligned}|L_2(u,v)| &\leqslant \max\left\{ 1, |\lambda|, 12|c_0|, 12|c_1|, -\frac{144\gamma_1}{\gamma_2}, \frac{144\delta_1}{\delta_2}, 12 \left| \frac{\alpha_2 \gamma_1 - \alpha_1 \gamma_2}{\gamma_2 \alpha_3} \right|, \right. \\ &\quad \left. 12 \left| \frac{\beta_2 \delta_1 - \beta_1 \delta_2}{\delta_2 \beta_3} \right| \right\} \|u\|_{\mathcal{K}} \|v\|_{\mathcal{K}}.\end{aligned}$$

As for the lower bound we get

$$\begin{aligned}\operatorname{Re} L_2(u,u) &= \lambda |||u|||^2 + \|u''\|_{L^2}^2 - \frac{\gamma_1}{\gamma_2} \left(|u'(0)|^2 + \frac{\alpha_2 \gamma_1 - \alpha_1 \gamma_2 + \gamma_0 \alpha_3}{\alpha_3 \gamma_1} \operatorname{Re}\bigl(u'(0)\overline{u(0)}\bigr) \right) \\ &\quad + c_0 |u(0)|^2 w_0 + \frac{\delta_1}{\delta_2} \left(|u'(1)|^2 + \frac{\beta_2 \delta_1 - \beta_1 \delta_2 + \delta_0 \beta_3}{\beta_3 \delta_1} \operatorname{Re}\bigl(u'(1)\overline{u(1)}\bigr) \right) \\ &\quad + c_1 |u(1)|^2 w_1 \end{aligned} \tag{49}$$

in the case $\gamma_1 \neq 0$, $\delta_1 \neq 0$. Let

$$k_0 = \frac{\alpha_2 \gamma_1 - \alpha_1 \gamma_2 + \gamma_0 \alpha_3}{\alpha_3 \gamma_1} \quad \text{and} \quad k_1 = \frac{\beta_2 \delta_1 - \beta_1 \delta_2 + \delta_0 \beta_3}{\beta_3 \delta_1}.$$

Then

$$\operatorname{Re} L_2(u,u) = \lambda |||u|||^2 + \|u''\|_{L^2}^2 - \frac{\gamma_1}{\gamma_2}\left[\left(\left|u'(0)\right|^2 + k_0 \operatorname{Re}\left(u'(0)\overline{u(0)}\right)\right) + \frac{k_0^2}{4}\left|u(0)\right|^2\right]$$
$$+ \frac{\delta_1}{\delta_2}\left[\left(\left|u'(1)\right|^2 + k_1 \operatorname{Re}\left(u'(1)\overline{u(1)}\right)\right) + \frac{k_1^2}{4}\left|u(1)\right|^2\right]$$
$$+ \left(c_0 - \frac{k_0^2\gamma_1\alpha_3}{4\gamma_2}\right)\left|u(0)\right|^2 w_0 + \left(c_1 - \frac{k_1^2\delta_1\beta_3}{4\delta_2}\right)\left|u(1)\right|^2 w_1.$$

Hence

$$\operatorname{Re} L_2(u,u) = \lambda |||u|||^2 + \|u''\|_{L^2}^2 - \frac{\gamma_1}{\gamma_2}\left|u'(0) + \frac{k_0}{2}u(0)\right|^2 + \frac{\delta_1}{\delta_2}\left|u'(1) + \frac{k_1}{2}u(1)\right|^2$$
$$+ \sum_{i=0}^{1} d_i \left|u(i)\right|^2 w_i,$$

where

$$d_0 = \frac{\alpha_2\gamma_0 - \alpha_0\gamma_2}{\gamma_2} - \frac{(\alpha_2\gamma_1 - \alpha_1\gamma_2 + \gamma_0\alpha_3)^2}{4\alpha_3\gamma_1\gamma_2}$$

and

$$d_1 = \frac{\beta_2\delta_0 - \beta_0\delta_2}{\delta_2} + \frac{(\beta_2\delta_1 - \beta_1\delta_2 + \delta_0\beta_3)^2}{4\beta_3\delta_1\delta_2}.$$

Choose $\lambda_0 > 2\max\{|d_0|, |d_1|\}$. Then if $\lambda > \lambda_0$, Re $L_2(u,u) \geqslant \epsilon_2 \|u\|_{\mathcal{K}}^2$ for some $\epsilon_2 > 0$; here we used $\frac{\gamma_1}{\gamma_2} \leqslant 0$ and $\frac{\delta_1}{\delta_2} \geqslant 0$. Again we can apply the Lax–Milgram Lemma to obtain a unique weak solution $u \in \mathcal{K}$ of

$$L_2(u,v) = F_2(v) \quad \text{for all } v \in \mathcal{K}$$

even if A is not symmetric so long as $\frac{\delta_1}{\delta_2} > 0$ and $\frac{\gamma_1}{\gamma_2} < 0$.

If $\delta_1 = 0$, we must have $\frac{\beta_1}{\beta_3} = \frac{\delta_0}{\delta_2}$; while if $\gamma_1 = 0$, we must have $\frac{\alpha_1}{\alpha_3} = \frac{\gamma_0}{\gamma_2}$. We give the argument for the case $\gamma_1 = 0$; the case of $\delta_1 = 0$ follows from a similar argument. When $\gamma_1 = 0, \operatorname{Re} L_2(u,u)$ takes the form

$$\operatorname{Re} L_2(u,u) = \lambda |||u|||^2 + \|u''\|_{L^2}^2 - \alpha_0\left|u(0)\right|^2 w_0$$
$$+ \frac{\delta_1}{\delta_2}\left(\left|u'(1)\right|^2 + \frac{\beta_2\delta_1 - \beta_1\delta_2 + \delta_0\beta_3}{\beta_3\delta_1}\operatorname{Re}\left(u'(1)\overline{u(1)}\right)\right) + c_1\left|u(1)\right|^2 w_1$$

since $\gamma_0\alpha_3 = \alpha_1\gamma_2$. In this case if we choose $\lambda_0 > 2\max\{|\alpha_0|, |d_1|\}$, we obtain the necessary lower bound.

In summary, in this section we have proved the following theorem.

Theorem 7. *Consider the operator $Au = u''''$ with boundary conditions* (2)–(5) *be acting on $D(A)$. Assume that $\delta_3 = \gamma_3 = 0$.*

(i) *If* either (17) *or* (20) *holds, then A is selfadjoint and bounded below if A is symmetric.*
(ii) *If* one *of* (18) *or* (19) *holds at $x = 0$* and one *of* (21) *or* (22) *holds at $x = 1$, then A is quasi-m-accretive.*

As a final comment, we note the following consequences for the beam equation.

Theorem 8. *Under the hypotheses of Theorem* 5*, the beam equation*

$$u_{tt} + c^2 u_{xxxx} = 0$$

with the boundary conditions of Theorem 5 *is well-posed and is governed by a strongly continuous cosine function on* X_2*.*

This theorem follows immediately from known results, cf. [11].

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